II. "Mathematical Contributions to the Theory of Evolution. IV. On the probable Errors of Frequency Constants and on the Influence of Random Selection on Variation and Correlation." By KARL PEARSON, M.A., F.R.S., Professor of Mathematics and Mechanics, University College, London, and L. N. G. FILON, B.A.

III. "On certain Media for the Cultivation of the Bacillus of Tubercle." By ARTHUR RANSOME, M.D., F.R.S.

IV. "Further Note on the Transplantation and Growth of Mammalian Ova within a Uterine Foster-mother." By WALTER HEAPE, M.A., Trinity College, Cambridge. Communicated by Dr. W. H. GASKELL, F.R.S.

V. "Further Observations upon the Comparative Physiology of the Suprarenal Capsules." By SWALE VINCENT, M.B. (Lond.), British Medical Association Research Scholar. Communicated by E. A. SCHÄFER, F.R.S.

VI. "Summary of Professor Edgeworth David's Preliminary Report on the Boring at Funafuti." By T. G. BONNEY, D.Sc., LL.D. (Univ. McGill), F.R.S., Professor of Geology in University College, London.

VII. "On the Determination of the Indices of Refraction of various Substances for the Electric Ray. II. Index of Refraction of Glass." By JAGADIS CHUNDER BOSE, M.A., D.Sc., Professor of Physical Science, Presidency College, Calcutta. Communicated by LORD RAYLEIGH, F.R.S.

VIII. "On the Influence of the Thickness of Air-space on Total Reflection of Electric Radiation." By JAGADIS CHUNDER BOSE, M.A., D.Sc., Professor of Physical Science, Presidency College, Calcutta. Communicated by Lord RAYLEIGH, F.R.S.

"On the Geometrical Treatment of the 'Normal Curve' of Statistics, with especial Reference to Correlation and to the Theory of Error." By W. F. SHEPPARD, M.A., LL.M., formerly Fellow of Trinity College, Cambridge. Communicated by Professor A. R. FORSYTH, F.R.S. Received October 9,—Read November 25, 1897.

(Abstract.)

The object of the paper is, in the first place, to simplify and extend the treatment of normal correlation as expounded by Francis Galton and Karl Pearson; and in the second place to obtain general

formulæ in the theory of error, and to apply them to questions which arise in relation to normal distributions and normal correlation. The method is, throughout, elementary, the use of the differential and integral calculus being avoided, though geometrical infinitesimals are to a certain extent employed.

Geometrical.

The *normal curve* is defined by the property that the product of the abscissa and the subtangent is constant; thus if MP is an ordinate from the base, and PT the tangent, OM . MT $= a^2$, O being a fixed point in the base. The area bounded by the curve and the base OMT is called a *normal figure*. The length $2a$ is the *parameter* of the figure. The definition of the curve leads at once to its projective properties, and also to the formula for the mean value of x^{2n+1} or x^{2n+2}, where x denotes the distance of an element of area from the central ordinate.

If the normal curve or normal figure is rotated about its central ordinate, it generates the normal surface or normal solid. It is proved geometrically that any vertical section of this solid (*i.e.*, any section by a plane perpendicular to its base plane) is a normal figure of the same parameter as the central sections; and, therefore, if the sections of the surface by any series of parallel vertical planes are projected on any plane of the series, the curves so obtained are orthogonal projections of one another with regard to their common base. The converse propositions are also established geometrically.

The volume of the solid is obtained in terms of its central ordinate and of the parameter of vertical sections; and thus it is found that the central ordinate of a normal figure of semi-parameter unity and area unity is $1/\sqrt{2\pi}$.

Let Σ be any closed curve in the base plane. Then it is shown how to construct a curve whose area shall be proportional to the portion of the solid which lies vertically above Σ, *i.e.*, to the volume which would be cut out of the solid by a cylinder having Σ for its cross-section. Thus, when Σ is given, this volume can be measured mechanically.

Statistical.

Let L and M denote the measures of two co-existent attributes, L_1 and M_1 their mean values, a^2 and b^2 the mean squares of the respective deviations from the means, and $ab \cos D$ the mean product of the deviations from the means. Then the angle D is called the *divergence*. The solid of frequency of $(L - L_1)/a \sin D$ and $(M - M_1)/b \sin D$, the planes of reference being inclined at an angle D to one another (so that the included angle is $180° - D$), is called the *correlation-solid*.

It is shown that, whatever the laws of distribution may be, the correlation-solid of the distributions of L and M is the same as that of the distributions of $l\mathrm{L}+m\mathrm{M}$ and $l'\mathrm{L}+m'\mathrm{M}$, where l, m, l', m' are any constants whatever.

If L and M are distributed "normally," and the distributions are independent, the correlation-solid will be a normal solid. Hence it follows that the distribution of $l\mathrm{L}+m\mathrm{M}$ is also normal.

Galton's definition of normal correlation is adopted; his "coefficient of correlation" being therefore cos D. It is shown that the correlation-solid of two normally correlated distributions is a normal solid, and, therefore, if the distributions of L and of M are normally correlated, the values of $l\mathrm{L}+m\mathrm{M}$ are normally distributed, and the distributions of $l\mathrm{L}+m\mathrm{M}$ and of $l'\mathrm{L}+m'\mathrm{M}$ are normally correlated.

The value of D, in a case of normal correlation, can be obtained without calculating the means, mean squares, and mean product. If we find the medians L_1 and M_1, and form a table of double classification, thus :—

	Below L_1.	Above L_1.
Below M_1.....	P	Q
Above M_1.....	Q	P

then $\mathrm{D} = \dfrac{\mathrm{Q}}{\mathrm{P}+\mathrm{Q}} \times 180°$.

If we know the proportions of individuals for which L exceeds values X and X′, and the proportions for which M exceeds values Y and Y′, we can, for any particular value of D, construct an area representing the proportion of individuals for which L lies between X and X′, and M between Y and Y′. The simplest case is that in which the distributions of L and of M are classified in the same way, *e.g.*, according to the "decile" method. The proportions of individuals falling into the 100 classes corresponding to a double decile classification are obtained by constructing a certain figure, which is the same whatever the value of D may be, and moving the figure through a distance equal to D/360° of its whole length. The diagram so obtained contains 100 areas, representing the proportions in the 100 classes in question.

The definitions of independence and of normal correlation are extended to any number of distributions, and it is shown that if the distributions of L, M, N.... are normal, and either independent or correlated, the values of $l\mathrm{L} + m\mathrm{M} + n\mathrm{N} + \ldots$ are normally distributed.

Theory of Error.

Let a community be divided into a number of classes, the proportions in the different classes being z_1, z_2, $z_3 \ldots$, so that $z_1 + z_2 + z_3 + \ldots = 1$. Suppose a random selection of n individuals to be made, the numbers drawn from the different classes being $n(z_1 + \epsilon_1)$, $n(z_2 + \epsilon_2)$, $n(z_3 + \epsilon_3) \ldots$ It is proved geometrically, with the aid of the binomial theorem, that the values of the errors ϵ_1, ϵ_2, $\epsilon_3 \ldots$ are normally distributed, and that the distributions are normally correlated. It follows that the values of any expression of the form $\Sigma A\epsilon = A_1\epsilon_1 + A_2\epsilon_2 + A_3\epsilon_3 + \ldots$ are normally distributed. The mean square of $\Sigma A\epsilon$ is shown to be $\{\Sigma A^2 z - (\Sigma Az)^2\} \div n$, and the mean product of $\Sigma A\epsilon$ and $\Sigma B\epsilon$ to be $\{\Sigma ABz - \Sigma Az \,.\, \Sigma Bz\} \div n$. The applications are of two kinds:—

(1) The values of the probable errors in the determination of certain quantities are obtained, and, in particular, the probable errors in the mean, mean square of deviation, mean product of deviations, and divergence.

(2) Formulæ are obtained for testing particular hypotheses; *e.g.*, whether two distributions (of any kind) are independent; whether a distribution is normal; and whether two normal distributions are correlated.

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(Abstract.)

1. This memoir starts with a general theorem, by which the probable errors made in calculating the constants of any frequency distribution may be determined. It is shown that these probable errors form a correlated system approximately following the normal law of frequency, whatever be the nature of the original frequency distribution, *i.e.*, whether it be skew or normal. The importance of this result for the theory of evolution is then drawn attention to. It is shown that any selection, whether of size, variation, or correlation, will in general involve a modification not only of the size, but the variation and correlation of the whole complex of correlated organs. The subject of directed selection, of which this random selection is only a special case, is reserved for another memoir, nearly completed.